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K symmetries and τ symmetries of evolution equations and their Lie algebras

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Abstract. On the basis of Tu's work, we establish a more general skeleton on K symmetries and τ symmetries of evolution equations and their Lie algebraic structures and discuss carefully the corresponding symmetries and Lie algebras for KdV and Jaulent-Miodek hierarchies.

1. Introduction

It is known (Chen *et al* 1982a, Li and Zhu 1987, 1986) that many integrable evolution equations possess a new set of symmetries, usually called τ symmetries, and these symmetries often constitute a Lie algebra together with the original symmetries, called K symmetries (for more information, see Chen *et al* 1982b, 1983, Zhu and Li 1987, 1986, Tian 1988, Li and Hu 1988). Furthermore, τ symmetries themselves are full of deep significance (Olver 1980, Chen *et al* 1987). Li and Cheng (Li and Cheng 1989, Cheng and Li 1987) found that there also exist new sets of symmetries for the evolution equations which take τ symmetries as vector fields. Recently Tu (1988) showed that these τ symmetries may be generated by the generators of the first degree (Fuchssteiner 1983). In this paper, we consider more general cases based upon Tu's work and give more infinite-dimensional Lie algebraic structures on K and τ symmetries. We explain that not all τ symmetries can form one Lie algebra with all K symmetries and discuss carefully two hierarchies of integrable evolution equations (KdV and Jaulent-Miodek hierarchies) as examples. In addition, we point out a few careless mistakes in the literature.

In the following, we give some basic notation, definitions and results. In accordance with Tu (1988), let $\mathbb{R}$ and $\mathbb{C}$ be the real and complex fields respectively, and let $\mathscr{S}$ be one linear topological space over $\mathbb{C}$. We denote by $\mathscr{L}$ all differentiable functions mapping $\mathbb{R}^N \times \mathbb{R} \times \mathscr{S}$ into $\mathscr{S}$.

Definition 1. Let $K = K(u) = K(x, t, u)$, $S = S(u) = S(x, t, u) \in \mathscr{L}$. The Gateaux derivative of $K(u)$ in the direction $S(u)$ with respect to u is defined by

$$K'[S] = K'(u)[S(u)] = \frac{\partial}{\partial \varepsilon} K(u + \varepsilon S(u))|_{\varepsilon=0}. \tag{1.1}$$

It is known that $\mathscr{L}$ forms a Lie algebra with respect to the following product:

$$[K, S] = [K(u), S(u)] = K'(u)[S(u)] - S'(u)[K(u)] \qquad K, S \in \mathscr{L}. \tag{1.2}$$

Assume that $u = u(x, t)$ is a differentiable function mapping $\mathbb{R}^N \times \mathbb{R}$ into $\mathcal{S}$. We consider some evolution equation

$$u_t = K(x, t, u) \qquad K \in \mathcal{L}. \tag{1.3}$$

Definition 2. A function $G = G(x, t, u) \in \mathcal{L}$ is called a symmetry of the equation (1.3) if G satisfies the linearised equation of (1.3)

$$\frac{\mathrm{d}G}{\mathrm{d}t} = K'(u)[G] \tag{1.4}$$

where $\mathrm{d}/\mathrm{d}t$ denotes the total t-derivative, u satisfies the equation (1.3) and $K'(u)[G]$ is defined as in (1.1).

Evidently, the linearised equation (1.4) is equivalent to the following equation (Fuchssteiner 1983):

$$\frac{\partial G}{\partial t} = [K, G] \tag{1.5}$$

where $[K, G]$ is defined as in (1.2). The symmetries defined in definition 2 are all infinitesimal generators of one-parameter groups of invariant transformations of equation (1.3).

Denote by $L(\mathcal{L})$ the linear operators mapping $\mathcal{L}$ into itself. Furthermore, denote by $\mathcal{U}$ the set of differentiable operators mapping $\mathbb{R}^n \times \mathbb{R} \times \mathcal{S}$ into $L(\mathcal{L})$ and suppose that $\Phi K = \Phi(x, t, u)K$ for $\Phi \in \mathcal{U}$, $K \in \mathcal{L}$, $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, $u \in \mathcal{S}$.

Definition 3. Let $\Phi \in \mathcal{U}$, $K \in \mathcal{L}$, then the Lie derivative (Magri 1980) $L_K\Phi \in \mathcal{U}$ of Φ with respect to K is defined by

$$(L_K\Phi)S = \Phi[K, S] - [K, \Phi S] \qquad S \in \mathcal{L}. \tag{1.6}$$

Definition 4. An operator $\Phi \in \mathcal{U}$ is called a hereditary symmetry (Fuchssteiner 1979, 1981) if the following holds:

$$\Phi^2[K, S] + [\Phi K, \Phi S] - \Phi\{[K, \Phi S] + [\Phi K, S]\} = 0 \qquad K, S \in \mathcal{L}. \tag{1.7}$$

An operator $\Phi \in \mathcal{U}$ is called a strong symmetry (or a recursion operator) of (1.3) if it maps one symmetry of (1.3) into another symmetry of (1.3).

It is easy to see that $\Phi \in \mathcal{U}$ is a hereditary symmetry if and only if

$$L_{\Phi K}\Phi = \Phi L_K \Phi \qquad K \in \mathcal{L} \tag{1.8}$$

and that $\Phi = \Phi(x, t, u) \in \mathcal{U}$ is a strong symmetry of (1.3) if and only if

$$\frac{\partial \Phi}{\partial t} + L_K \Phi = 0. \tag{1.9}$$

In addition, if we define

$$\Phi'[K]S = \frac{\partial}{\partial \varepsilon}\Phi(u + \varepsilon K)S|_{\varepsilon=0} \qquad S \in \mathcal{L}$$

then it is easily proved that

$$L_K\Phi = \Phi'[K] - [K', \Phi] = \Phi'[K] - K'\Phi + \Phi K' \tag{1.10}$$

and that (1.7) is equivalent to (Fuchssteiner and Fokas 1981)

$$\Phi'[\Phi K]S - \Phi'[\Phi S]K - \Phi\{\Phi'[K]S - \Phi'[S]K\} = 0 \qquad K, S \in \mathscr{L}. \tag{1.11}$$

The strong symmetries and hereditary symmetries play a crucial role when generating K symmetries and τ symmetries of evolution equations in our theoretical skeleton.

2. Some fundamental lemmas

In this section we shall present several basic lemmas which are important for finding Lie algebraic structure of K symmetries and τ symmetries for evolution equations.

Let $\Phi \in \mathscr{U}$. We accept $0\Phi^{-1} = 0$, and suppose that Φ satisfies that $\Phi\{f(t)K\} = f(t)\Phi K$ for $K \in \mathscr{L}$ and an arbitrary differentiable time function $f: \mathbb{R} \to \mathbb{C}$ and that α, β, γ, δ are all polynomials of Φ with time-function coefficients, for example $\alpha = \alpha(\Phi) = \sum \alpha_i \Phi^i = \sum \alpha_i(t)\Phi^i$.

Lemma 1. Let $\Phi \in \mathscr{U}$ be a hereditary symmetry and $L_K\Phi = \alpha = \alpha(\Phi)$ for some $K \in \mathscr{L}$, then

$$L_{\Phi^m K}\Phi^n = n\alpha\Phi^{m+n-1} \qquad m, n \geqslant 0. \tag{2.1}$$

Proof. When $n = 0$, $L_S\Phi^n = L_S I = 0$, for $S \in \mathscr{L}$. Thus (2.1) holds for $n = 0$.

In the following, we let $n \geqslant 1$. By (1.8)

$$L_{\Phi^m K}\Phi = \Phi^m L_K \Phi = \alpha\Phi^m \qquad m \geqslant 0. \tag{2.2}$$

When $n > 1$, by the chain rule for derivatives of operators and (2.2) we have

$$\begin{aligned}(\Phi^n)'[\Phi^m K] &= \sum_{i=1}^{n} \Phi^{n-i}\Phi'[\Phi^m K]\Phi^{i-1} \\ &= \sum_{i=1}^{n} \Phi^{n-i}(L_{\Phi^m K}\Phi + [(\Phi^m K)', \Phi])\Phi^{i-1} \\ &= n\alpha\Phi^{m+n-1} + [(\Phi^m K)', \Phi^n] \qquad m \geqslant 0.\end{aligned}$$

Thus

$$L_{\Phi^m K}\Phi^n = n\alpha\Phi^{m+n-1} \qquad m \geqslant 0, n \geqslant 1$$

which completes the proof. □

From this lemma, we can obtain the two following corollaries at once.

Corollary 1. Let $\Phi \in \mathscr{U}$ be hereditary. Then if $L_K\Phi = 0$ for some $K \in \mathscr{L}$, we have

$$[\Phi^m K, \Phi^n S] = \Phi^n[\Phi^m K, S] \qquad S \in \mathscr{L}, m, n \geqslant 0. \tag{2.3}$$

In particular,

$$[\Phi^m K, \Phi^n K] = 0 \qquad m, n \geqslant 0.$$

Corollary 2. Let $\Phi \in \mathscr{U}$ be hereditary. If $\Phi = \Phi(x, u)$ is a strong symmetry of (1.3), then the Φ^n, $n \geqslant 0$, are all strong symmetries of the equation $u_t = \Phi^l K (l \geqslant 0)$.

Lemma 2. Let $\Phi \in \mathscr{U}$ be a hereditary symmetry, $K, S \in \mathscr{L}$, and $L_K\Phi = \alpha = \alpha(\Phi)$, $L_S\Phi = \beta = \beta(\Phi)$. Then

$$[\Phi^{m+k}K, \Phi^{n+l}S] = \Phi^{m+n}\{[\Phi^k K, \Phi^l S] + (m\beta\Phi^{k+l-1}K - n\alpha\Phi^{k+l-1}S)\} \qquad m, n, k, l \geqslant 0. \tag{2.4}$$

Proof. By (1.6), for $m, n, k, l \geqslant 0$, we have

$$[\Phi^{m+k}K, \Phi^{n+l}S] = -(L_{\Phi^{m+k}K}\Phi^n)\Phi^l S + \Phi^n[\Phi^{m+k}K, \Phi^l S]$$

$$[\Phi^{m+k}K, \Phi^l S] = (L_{\Phi^l S}\Phi^m)\Phi^k K + \Phi^m[\Phi^k K, \Phi^l S].$$

The desired result follows these by lemma 1. □

It is easy to deduce the following.

Corollary 3. Let $\Phi \in \mathscr{U}$ be hereditary, $K \in \mathscr{L}$, and $L_K\Phi = \alpha$, then

$$[\Phi^m K, \Phi^n K] = \alpha(m-n)\Phi^{m+n-1}K \qquad m, n \geqslant 0. \tag{2.5}$$

In addition to the assumptions of corollary 3, let $\alpha = 0$ and $\partial\Phi/\partial t = 0$, $\partial K/\partial t = 0$, i.e. $\Phi = \Phi(x, u)$, $K = K(x, u)$, this moment Φ is a strong symmetry of (1.3) by (1.9). Then the evolution equations $u_t = \Phi^l K$, $l \geqslant 0$, possess a hierarchy of common symmetries $\{\Phi^m K\}_{m=0}^{\infty}$ which commute with each other. For many hierarchies of soliton equations, this is usually verified by using the Hamiltonian structures of the hierarchies. But, here we see that that point can also be deduced directly from the strong and hereditary properties.

Lemma 3. Let $\Phi \in \mathscr{U}$ be a hereditary symmetry, $K, S \in \mathscr{L}$, and l be a non-negative integer. If $L_K\Phi = \alpha = \alpha(\Phi)$, $L_S\Phi = \beta = \beta(\Phi)$ and $\Phi^l[K, S] = \gamma K + \delta S = \gamma(\Phi)K + \delta(\Phi)S$, then $\{\Phi^{m+l}K, \Phi^{n+l}S | m, n \geqslant 0\}$ constitute an infinite-dimensional Lie algebra and possess the following commutator relations:

$$[\Phi^{m+l}K, \Phi^{n+l}K] = \alpha(m-n)\Phi^{m+n+2l-1}K \tag{2.6}$$

$$\begin{aligned}[\Phi^{m+l}K, \Phi^{n+l}S] &= \gamma\Phi^{m+n+l}K + \delta\Phi^{m+n+l}S + (m+l)\beta\Phi^{m+n+2l-1}K \\ &\quad - (n+l)\alpha\Phi^{m+n+2l-1}S\end{aligned} \tag{2.7}$$

$$[\Phi^{m+l}S, \Phi^{n+l}S] = \beta(m-n)\Phi^{m+n+2l-1}S \tag{2.8}$$

where $m, n \geqslant 0$.

Proof. The relations (2.6), (2.8) are easily deduced from corollary 3. By (2.4), for $m, n \geqslant 0$ we have

$$\begin{aligned}[\Phi^{m+l}K, \Phi^{n+l}S] &= \Phi^{m+n+2l}[K, S] + (m+l)\beta\Phi^{m+n+2l-1}K - (n+l)\alpha\Phi^{m+n+2l-1}S \\ &= \gamma\Phi^{m+n+l}K + \delta\Phi^{m+n+l}S + (m+l)\beta\Phi^{m+n+2l-1}K - (n+l)\alpha\Phi^{m+n+2l-1}S.\end{aligned}$$

Thus the relation (2.7) also holds for $m, n \geqslant 0$. It follows from (2.6)-(2.8) that $\{\Phi^{m+l}K, \Phi^{n+l}S | m, n \geqslant 0\}$ form an infinite-dimensional Lie algebra. □

Lemma 4. Let $\Phi \in \mathscr{U}$ be a hereditary symmetry, $K, S \in \mathscr{L}$, and l be a non-negative integer. If $L_K\Phi = 0$, $L_S\Phi = \beta = \beta(\Phi)$ and $\Phi^l[K, S] = \gamma K = \gamma(\Phi)K$, then $\{\Phi^m K, \Phi^{n+l}S | m, n \geq 0\}$ constitute an infinite-dimensional Lie algebra and possess the following commutator relations:

$$[\Phi^m K, \Phi^n K] = 0 \tag{2.9}$$

$$[\Phi^m K, \Phi^{n+l} S] = \gamma \Phi^{m+n} K + m\beta \Phi^{m+n+l-1} K \tag{2.10}$$

$$[\Phi^{m+l} S, \Phi^{n+l} S] = \beta(m-n)\Phi^{m+n+2l-1} S \tag{2.11}$$

where $m, n \geq 0$.

The proof, which is similar to the proof of lemma 3, is omitted.

In order to derive τ symmetries of evolution equations, we shall make use of one result of Fuchssteiner's.

Lemma 5 (Fuchssteiner 1983). Let $K = K(x, u)$, $S = S(x, u) \in \mathscr{L}$. If $[K, [K, S]] = 0$, then $\tau = t[K, S] + S$ is a symmetry of (1.3).

This lemma shows that if $S = S(x, u) \in \mathscr{L}$ is the first-degree generator of the equation $u_t = K(x, u)$, then we obtain a first-degree time-dependent symmetry $\tau = t[K, S] + S$ of the same equation.

3. *K* symmetries, τ symmetries and their Lie algebras

In this section, we shall give a sufficient condition under which a given evolution equation possesses K symmetries and τ symmetries, and establish the Lie algebraic structures of the K symmetries and τ symmetries for the equation.

Theorem 1. Let $\Phi = \Phi(x, u) \in \mathscr{U}$ be a hereditary symmetry, $K = K(x, u)$, $S = S(x, u) \in \mathscr{L}$, and let l, p be two non-negative integers. Suppose that $L_K\Phi = 0$, $L_S\Phi = \beta = \beta(\Phi)$, and $\Phi^l[K, S] = \gamma K = \gamma(\Phi)K$, then the evolution equation

$$u_t = K_p = \Phi^p K \tag{3.1}$$

possesses two hierarchies of symmetries $\{K_m = \Phi^m K\}_{m=0}^{\infty}$ and $\{\tau_n^p = \Phi^n \tau^p\}_{n=0}^{\infty}$, where τ^p is defined by

$$\tau^p = \begin{cases} \tau^{pl} = t[K_p, \Phi^{l-p} S] + \Phi^{l-p} S = t(\gamma K + p\beta\Phi^{l-1} K) + \Phi^{l-p} S & \text{for } p < l \\ \tau^{lp} = t[K_p, S] + S = t(\gamma\Phi^{p-l} K + p\beta\Phi^{p-1} K) + S & \text{for } p \geq l. \end{cases} \tag{3.2}$$

Proof. By using lemma 2, we have

$$[\Phi^m K, \Phi^n K] = 0 \qquad m, n \geq 0 \tag{3.3}$$

$$[\Phi^m K, \Phi^n S] = \gamma\Phi^{m+n-l} K + m\beta\Phi^{m+n-1} K \qquad m, n \geq 0, m+n \geq l \tag{3.4}$$

$$[\Phi^m S, \Phi^n S] = \beta(m-n)\Phi^{m+n-1} S \qquad m, n \geq 0. \tag{3.5}$$

Noticing $\partial K_m/\partial t = 0$, $m \geq 0$, it follows from (3.3) that $\{K_m\}_{m=0}^{\infty}$ is a hierarchy of symmetries of (3.1). Based on (3.3), (3.4), we can obtain that

$$[K_p, [K_p, \Phi^{l-p} S]] = 0 \qquad \text{for } p < l$$

$$[K_p, [K_p, S]] = 0 \qquad \text{for } p \geq l.$$

Hence by lemma 5, τ^p is a symmetry of (3.1). In addition, from corollary 2, we know that the Φ^n, $n \geqslant 0$, are all strong symmetries of (3.1). Therefore the $\tau_n^p = \Phi^n \tau^p$, $n \geqslant 0$, are all symmetries of (3.1). The formula (3.2) is a direct corollary of (3.3), (3.4). □

Remark 1. We require that $\partial\Phi/\partial t = 0$, $\partial K/\partial t = 0$ in the theorem. If we do not have this condition, then the $\Phi^m K$, $m \geqslant 0$, are not certain to be symmetries of (3.1).

Remark 2. By lemma 1 and corollary 1, we easily obtain the following:

$$L_{\Phi^n S}\Phi = \beta\Phi^n \qquad \Phi^k[K, \Phi^n S] = \gamma\Phi^{n+k-l}K \qquad n, k \geqslant 0, n+k \geqslant l.$$

Thus the $\Phi^n S$, $n \geqslant 1$, all possess the property which S satisfies. But we should not obtain other new symmetries of (3.1) beginning with $\Phi^n S (n \geqslant 1)$.

Corollary 4. Let the assumption be the same as that of theorem 1. When $p < l$, equation (3.1) possesses symmetries $\{[\Phi^i K, \Phi^j S] | 0 \leqslant i+j \leqslant l-1, j \geqslant l-p\}$; when $p \geqslant l$, equation (3.1) possesses symmetries $\{[\Phi^i K, \Phi^j S] | 0 \leqslant i+j \leqslant l-1\}$.

Proof. When $p < l$, for $i, j \geqslant 0$ satisfying $j \geqslant l-p$, $0 \leqslant i+j \leqslant l-1$

$$[\Phi^i K, \Phi^j S] = \Phi^{j-l+p}[\Phi^i K, \Phi^{l-p} S] = \Phi^{j-l+p}[\Phi^i K, \tau_0^p].$$

When $p \geqslant l$, for $i, j \geqslant 0$ satisfying $0 \leqslant i+j \leqslant l-1$

$$[\Phi^i K, \Phi^j S] = \Phi^j[\Phi^i K, S] = \Phi^j[\Phi^i K, \tau_0^p].$$

By also noting corollary 2 and theorem 1, we obtain the desired results. □

Theorem 2. Under the assumption of theorem 1, set

$$\tau_n = \begin{cases} \Phi^{n+p}\tau^{pl} & \text{for } p < l,\ n \geqslant 0 \\ \Phi^{n+l}\tau^{lp} & \text{for } p \geqslant l,\ n \geqslant 0. \end{cases} \tag{3.6}$$

Then two hierarchies of symmetries $\{K_m\}_{m=0}^{\infty}$ and $\{\tau_n\}_{n=0}^{\infty}$ constitute an infinite-dimensional Lie subalgebra of $\mathscr{L}$ and possess the following commutator relations:

$$[K_m, K_n] = 0 \tag{3.7}$$

$$[K_m, \tau_n] = \gamma K_{m+n} + m\beta K_{m+n+l-1} \qquad K_{-1} = 0 \tag{3.8}$$

$$[\tau_m, \tau_n] = \beta(m-n)\tau_{m+n+l-1} \qquad \tau_{-1} = 0 \tag{3.9}$$

where $m, n \geqslant 0$.

Proof. Relation (3.7) is just (3.3), which has been verified. Now we prove relation (3.8). When $p < l$, we have, by using (3.3), (3.4),

$$\begin{aligned}[K_m, \tau_n] &= [\Phi^m K, t[\Phi^p K, \Phi^{n+l} S] + \Phi^{n+l} S] \\ &= [\Phi^m K, \Phi^{n+l} S] = \gamma K_{m+n} + m\beta K_{m+n+l-1} \qquad \text{for } m, n \geqslant 0.\end{aligned}$$

When $p \geqslant l$, similarly we have

$$\begin{aligned}[K_m, \tau_n] &= [\Phi^m K, t[\Phi^p K, \Phi^{n+l} S] + \Phi^{n+l} S] \\ &= [\Phi^m K, \Phi^{n+l} S] = \gamma K_{m+n} + m\beta K_{m+n+l-1} \qquad \text{for } m, n \geqslant 0.\end{aligned}$$

Thus the relation (3.8) holds for $m, n \geqslant 0$. Next we shall prove relation (3.9). Let $p < l$. Notice that we have (3.3)-(3.5) and $\Phi\{f(t)K\} = f(t)\Phi K$ for $f: \mathbb{R} \to \mathbb{C}$.

$$[\tau_m, \tau_n] = [t(\gamma\Phi^{m+p}K + p\beta\Phi^{m+p+l-1}K) + \Phi^{m+l}S, t(\gamma\Phi^{n+p}K + p\beta\Phi^{n+p+l-1}K) + \Phi^{n+l}S]$$

$$= t\left[\sum \gamma_i \Phi^{m+p+i} K + p \sum \beta_i \Phi^{m+p+l+i-1} K, \Phi^{n+l} S\right]$$

$$-(\text{exchange term of } m, n) + [\Phi^{m+l} S, \Phi^{n+l} S]$$

$$= t\Big(\sum \gamma_i \Phi^i [\gamma \Phi^{m+n+p} K + (m+p+i)\beta \Phi^{m+n+p+l-1} K]$$

$$+ p \sum \beta_i \Phi^i [\gamma \Phi^{m+n+p+l-1} K + (m+p+l+i-1)\beta \Phi^{m+n+p+2l-2} K]\Big)$$

$$-(\text{exchange term of } m, n) + \beta(m-n)\Phi^{m+n+2l-1} S$$

$$= t(m-n)\beta\gamma \Phi^{m+n+p+l-1} K + tp(m-n)\beta^2 \Phi^{m+n+p+2l-2} K$$

$$+ \beta(m-n)\Phi^{m+n+2l-1} S$$

$$= \beta(m-n)\Phi^{m+n+p+l-1}\tau^p = \beta(m-n)\tau^p_{m+n+p+l-1}$$

$$= \beta(m-n)\tau_{m+n+l-1} \qquad \text{for } m, n \geqslant 0.$$

When $p \geqslant l$, the proof is completely similar. Thus (3.9) holds also for $m, n \geqslant 0$. This completes the proof. □

Especially when $l = 0$, 1 or $p = 0$, we obtain the following two results from theorems 1 and 2.

Corollary 5. Let $\Phi = \Phi(x, u) \in \mathscr{U}$ be hereditary, $K = K(x, u)$, $S = S(x, u) \in \mathscr{L}$ and suppose that $L_K \Phi = 0$, $L_S \Phi = \beta$.

(i) When $[K, S] = \gamma K$, equation (3.1) possesses symmetries

$$K_m = \Phi^m K \qquad m \geqslant 0$$

$$\tau^p_n = t(\gamma \Phi^{n+p} K + p\beta \Phi^{n+p-1} K) + \Phi^n S \qquad n \geqslant 0.$$

Furthermore, $\{K_m\}_{m=0}^{\infty}$ and $\{\tau_n = \tau^p_n\}_{n=0}^{\infty}$ constitute a Lie subalgebra of $\mathscr{L}$ and satisfy

$$[K_m, K_n] = 0 \qquad m, n \geqslant 0$$

$$[K_m, \tau_n] = \gamma K_{m+n} + m\beta K_{m+n-1} \qquad m, n \geqslant 0$$

$$[\tau_m, \tau_n] = \beta(m-n)\tau_{m+n-1} \qquad m, n \geqslant 0.$$

(ii) When $\Phi[K, S] = \gamma K$, equation (3.1) possesses symmetries

$$K_m = \Phi^m K \qquad m \geqslant 0$$

$$\tau^p_n = \begin{cases} t\gamma \Phi^n K + \Phi^{n+1} S & p = 0, n \geqslant 0 \\ t(\gamma + p\beta)\Phi^{n+p-1} K + \Phi^n S & p \geqslant 1, n \geqslant 0. \end{cases}$$

Furthermore, setting

$$\tau_n = \begin{cases} t\gamma \Phi^n K + \Phi^{n+1} S & p = 0, n \geqslant 0 \\ t(\gamma + p\beta)\Phi^{n+p} K + \Phi^{n+1} S & p \geqslant 1, n \geqslant 0 \end{cases}$$

$\{K_m\}_{m=0}^{\infty}$ and $\{\tau_n\}_{n=0}^{\infty}$ constitute a Lie subalgebra of $\mathscr{L}$ and satisfy

$$[K_m, K_n] = 0 \qquad m, n \geqslant 0$$

$$[K_m, \tau_n] = (\gamma + m\beta) K_{m+n} \qquad m, n \geqslant 0$$

$$[\tau_m, \tau_n] = \beta(m-n)\tau_{m+n} \qquad m, n \geqslant 0.$$

Corollary 6. Let $\Phi=\Phi(x,u)\in\mathscr{U}$ be hereditary, $K=K(x,u)$, $S=S(x,u)\in\mathscr{L}$ and suppose that $L_K\Phi=0$, $L_S\Phi=\beta$, $\Phi'[K,S]=\gamma K$. Then the equation $u_t=K(x,u)$ possesses symmetries

$$K_m=\Phi^m K \qquad m\geqslant 0$$

$$\tau_n^0=t\gamma\Phi^n K+\Phi^{n+l}S \qquad n\geqslant 0$$

which constitute a Lie subalgebra of $\mathscr{L}$ and satisfy

$$[K_m,K_n]=0 \qquad m,n\geqslant 0$$

$$[K_m,\tau_n^0]=\gamma K_{m+n}+m\beta K_{m+n+l-1} \qquad m,n\geqslant 0$$

$$[\tau_m^0,\tau_n^0]=\beta(m-n)\tau_{m+n+l-1}^0 \qquad m,n\geqslant 0.$$

If we set $\tau_n=\tau_n^p$, $n\geqslant 0$, then $\{K_m\}_{m=0}^{\infty}$ and $\{\tau_n\}_{n=0}^{\infty}$ form a Lie algebra by the above two corollaries when $p=0$ or $l=0$, but in general, do not form a Lie algebra when $p>0$ and $l>0$. When $l=1$ and $p=1$, the result (ii) of corollary 5 corresponds to the main result (theorem 1) of Tu (1988) and shows that $\{K_m=\Phi^m K\}_{m=0}^{\infty}$ and $\{\tau_n=\tau_{n+1}^1=\Phi^{n+1}\tau^{11}\}_{n=0}^{\infty}$ form a Lie algebra with respect to the Lie product (1.2). But, since $[K,\tau_0^1]=[K,S]$ is not generally a linear combination of the K_m, $m\geqslant 0$, and the τ_n, $n\geqslant 0$, $\{K_m=\Phi^m K\}_{m=0}^{\infty}$ and $\{\tau_n=\tau_{n+1}^1=\Phi^{n+1}\tau^{11}\}_{n=-1}^{\infty}$ are not certain to constitute a Lie algebra. In theorem 1 of Tu (1988), the description of this is not completely correct.

4. Applications to integrable evolution equations

The theoretical skeleton proposed in the last section may be applied to a large number of hierarchies of integrable evolution equations. In the following, we shall discuss only two hierarchies of integrable evolution equations (KdV and Jaulent–Miodek hierarchies) as examples.

4.1. The KdV hierarchy

Let us first consider KdV hierarchy

$$u_t=K_p=\Phi^p u_x \qquad x,t\in\mathbb{R} \qquad p\geqslant 0 \qquad \Phi=\mathrm{D}^2+4u+2u_x\mathrm{D}^{-1} \qquad \mathrm{D}=\frac{\mathrm{d}}{\mathrm{d}x}. \tag{4.1}$$

Choosing $K=u_x$, $S=\frac{1}{2}$, we have

$$L_K\Phi=0 \qquad L_S\Phi=2 \qquad \Phi[K,\tfrac{1}{2}]=[K,\Phi(\tfrac{1}{2})]=[u_x,2u+xu_x]=K.$$

Note $l=1$ in this case. We obtain by lemma 2 that

$$[\Phi^m K,\Phi^n K]=0 \qquad m,n\geqslant 0 \tag{4.2}$$

$$[\Phi^m K,\Phi^n S]=(2m+1)\Phi^{m+n-1}K \qquad m,n\geqslant 0,\ m+n\geqslant 1 \tag{4.3}$$

$$[\Phi^m S,\Phi^n S]=2(m-n)\Phi^{m+n-1}S \qquad m,n\geqslant 0. \tag{4.4}$$

From corollary 5 we obtain two hierarchies of symmetries for the KdV equation of order p, i.e. $u_t=K_p(p\geqslant 1)$

$$K_m=\Phi^m u_x \qquad m\geqslant 0$$

$$\tau_n^p=\Phi^n\tau^p=\Phi^n(t[K_p,\tfrac{1}{2}]+\tfrac{1}{2})=(2p+1)tK_{n+p-1}+\Phi^n(\tfrac{1}{2}) \qquad n\geqslant 0.$$

Set $\tau_n = \Phi^{n+l}\tau^p = \Phi^{n+1}\tau^p = \tau^p_{n+1}$, $n \geq -1$. Then by corollary 5 we see that $\{K_m\}_{m=0}^{\infty}$ and $\{\tau_n\}_{n=0}^{\infty}$ constitute a Lie algebra

$$[K_m, K_n] = 0 \tag{4.5}$$

$$[K_m, \tau_n] = (2m+1)K_{m+n} \qquad K_{-1} = 0 \tag{4.6}$$

$$[\tau_m, \tau_n] = 2(m-n)\tau_{m+n} \qquad \tau_{-2} = 0. \tag{4.7}$$

By using (4.2)-(4.4), it is easy to show that (4.6), (4.7) hold for $n = -1$. Therefore $\{K_m\}_{m=0}^{\infty}$ and $\{\tau_n\}_{n=-1}^{\infty}$ also constitute a Lie algebra whose commutator relations are given by (4.5)-(4.7). Here we have obtained

$$[K_0, \tau_n] = [K_0, \Phi^{n+1}(\tfrac{1}{2})] = K_n \qquad n \geq -1$$

which makes up the deficiency of Lie algebraic structure of symmetries in Li and Zhu (1987). The same deficiency appears in Li and Zhu (1986).

In addition, by corollary 6, we obtain a hierarchy of symmetries for the equation $u_t = u_x$

$$\tau^0_n = \tau^0_n(x, t, u) = tK_n + \Phi^{n+1}(\tfrac{1}{2}) \qquad n \geq 0.$$

Here the τ^0_n, $n \geq 1$, are all non-local, but $\tau^0_0 = tu_x + 2u + xu_x$ is local. Setting $\tau_n = \tau^0_n$ for $n \geq 0$, then the two hierarchies of symmetries $\{K_m\}_{m=0}^{\infty}$ and $\{\tau_n\}_{n=0}^{\infty}$ constitute a Lie algebra and also satisfy the relations (4.5)-(4.7).

4.2. The Jaulent–Miodek hierarchy

Next we consider Jaulent–Miodek hierarchy (Jaulent and Miodek 1977)

$$u_t = K_p = \Phi^p u_x \qquad x, t \in \mathbb{R} \qquad p \geq 0 \tag{4.8}$$

where

$$u = \begin{bmatrix} q \\ r \end{bmatrix} \qquad \Phi = \begin{bmatrix} 0 & -\frac{1}{4}D^2 + q + \frac{1}{2}q_x D^{-1} \\ 1 & r + \frac{1}{2}r_x D^{-1} \end{bmatrix} \qquad D = \frac{d}{dx}.$$

Choose

$$K = u_x = \begin{bmatrix} q_x \\ r_x \end{bmatrix} \qquad S = \begin{bmatrix} -\frac{1}{2}r \\ 1 \end{bmatrix}.$$

Then we obtain

$$L_K \Phi = 0 \qquad L_S \Phi = \tfrac{1}{2} \qquad \Phi[K, S] = [K, \Phi S] = \tfrac{1}{2}K.$$

Similarly we have

$$[\Phi^m K, \Phi^n K] = 0 \qquad m, n \geq 0 \tag{4.9}$$

$$[\Phi^m K, \Phi^n S] = \tfrac{1}{2}(m+1)\Phi^{m+n-1}K \qquad m, n \geq 0, m+n \geq 1 \tag{4.10}$$

$$[\Phi^m S, \Phi^n S] = \tfrac{1}{2}(m-n)\Phi^{m+n-1}S \qquad m, n \geq 0. \tag{4.11}$$

Thus from corollary 5 we obtain two hierarchies of symmetries for the Jaulent–Miodek equation of order p, i.e. $u_t = K_p (p \geq 1)$

$$K_m = \Phi^m u_x \qquad m \geq 0$$

$$\tau^p_n = \Phi^n \tau^p = \Phi^n(t[K_p, S] + S) = \tfrac{1}{2}(p+1)tK_{n+p-1} + \Phi^n S \qquad n \geq 0.$$

Let $\tau_n = \Phi^{n+1}\tau^p = \tau^p_{n+1}$, $n \geqslant -1$, then, also by corollary 5, we know that $\{K_m\}_{m=0}^{\infty}$ and $\{\tau_n\}_{n=0}^{\infty}$ constitute a Lie algebra

$$[K_m, K_n] = 0 \tag{4.12}$$

$$[K_m, \tau_n] = \tfrac{1}{2}(m+1)K_{m+n} \qquad K_{-1} = 0 \tag{4.13}$$

$$[\tau_m, \tau_n] = \tfrac{1}{2}(m-n)\tau_{m+n} \qquad \tau_{-2} = 0. \tag{4.14}$$

By (4.9)-(4.11), when $n=-1$, (4.13) and (4.14) hold. Thus $\{K_m\}_{m=0}^{\infty}$ and $\{\tau_n\}_{n=-1}^{\infty}$ also constitute a Lie algebra whose Lie algebraic structure is still given by (4.12)-(4.14).

Besides, similarly by corollary 6, we can obtain the following two hierarchies of symmetries for the equation $u_t = u_x$

$$K_m = \Phi^m u_x \qquad m \geqslant 0$$

$$\tau_n = \tau_n^0 = \tfrac{1}{2}tK_n + \Phi^{n+1}S \qquad n \geqslant 0$$

in particular,

$$\tau_0 = \begin{bmatrix} q + \frac{1}{2}tq_x + \frac{1}{2}xq_x \\ \frac{1}{2}r + \frac{1}{2}tr_x + \frac{1}{2}xr_x \end{bmatrix}.$$

Those two hierarchies of symmetries constitute a Lie algebra whose Lie algebraic structure is also given by (4.12)-(4.14).

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